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A RUNOFF HYDROGRAPH EQUATION

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CONTENTS

	Page
Introduction	3
Derivation of the equation	4
Appendix	12

A RUNOFF HYDROGRAPH EQUATION [1]

Donn G. DeCoursey [2]

INTRODUCTION

This paper presents an equation that defines the surface runoff hydrograph and develops a method for determining the constants in the equation. It was derived in studies of six watersheds in the Oklahoma portion of the Washita River Basin. The equation was developed as one part of a comprehensive research program to determine the changes in flow regimen along the main stem of the Washita River associated with the application of flood control and watershed management measures on upstream tributaries.

It is believed that the equation fills the need for a general functional relation that can be used in the electronic computer analysis of streamflow problems. Many conceptual models or rational approaches have been proposed for delineating the runoff hydrograph [3] and some of these concepts have been used in developing this equation. The emphasis here, however, has been placed on having the equation related to various hydrograph properties.

1/ Soil and Water Conservation Research Division, Agricultural Research Service, USDA, in cooperation with the Oklahoma Agricultural Experiment Station.

2/ Hydraulic engineer (Research), USDA, Chickasha, Okla.

3/ Chow, Ven Te. 1964. Runoff. Handbook of Applied Hydrology. Section 14, pp. 14-27 to 14-35. McGraw Hill Book Co., New York.

DERIVATION OF THE EQUATION

Perhaps the best single equation for representing a hydrograph was derived by Edson[4/] and discussed by Bloomsburg[5/];

$$q = a\ t^b\ e^{-ct} \qquad [1]$$

where q is discharge in cubic feet per second, t is time in hours from beginning of the rise, e is the base of natural logarithms, and a, b, and c are constants.

The equation is based in part on the assumption that the cumulation of drainage area contributing to runoff is proportional to some power of time;

$$A \propto t^b \qquad b > 1.0 \qquad [2]$$

If runoff is expressed in watershed surface inches, q will also be proportional to some power of time;

$$q \propto t^b \qquad b > 1.0 \qquad [3]$$

A third assumption is that the reservoir action of the stream causes the discharge to decrease exponentially:

$$q \propto e^{-ct} \qquad c > 1.0 \qquad [4]$$

Since a variable which is proportional to two functions is also proportional to their product:

$$q \propto t^b\ e^{-ct}$$

With a as the proportionality constant, the equation becomes:

$$q = a\ t^b\ e^{-ct} \qquad [1]$$

4/ Edson, C. G. August 1951. Parameters for relating unit hydrographs to watershed characteristics. Trans. Amer. Geophys. Union, 32(4): 591-596.

5/ Bloomsburg, G. L. March 30, 1960. A hydrograph equation. Agricultural Research Service, SWCRD Research Report No. 325, 13 pp.

The constants in this equation may be solved for a particular watershed in terms of its hydrograph characteristics; time to peak, peak discharge, and the volume of runoff. This was done for the hydrographs of six watersheds. The major features of the six hydrographs used for the analysis are presented in table 1[6/]. Figure 1[6/] shows the hydrograph for Rock Creek and the hydrograph as obtained from the equation.

The recession side of the equation was depleted too quickly. This was characteristic of all six watersheds. Therefore, it was decided to combine the above equation with a second equation that would define the slope of the recession. The equation used for the recession was:

$$q = q_o \, e^{-\frac{t-t_o}{d}} \qquad [5]$$

in which q is the discharge in cubic feet per second at time t; q_o is the discharge at time t_o; e is the base of natural logarithms; and d is a recession constant.

Equation 5 may be rewritten by taking the natural logarithm of both sides:

$$\ln q = \ln q_o - \frac{t-t_o}{d}$$

or

$$\ln q = \ln q_o + \frac{t_o}{d} - \frac{t}{d}$$

Since q_o and t_o are constants in the equation; ln q is the dependent variable, $\ln q_o + \frac{t_o}{d}$ is the dependent variable intercept for t equal to zero, t is the independent variable, and $-\frac{1}{d}$ is the slope of the line. For a given watershed, the numerical value of d may be obtained by plotting the normal recession for the surface runoff hydrograph in semilogarithmic form and drawing a straight line through the results. The slope of this line is $-\frac{1}{d}$.

6/ All tables and figures are presented in the appendix.

By combining equations 1 and 5 and integrating, the total volume of runoff during the period of the hydrograph is given by:

$$Q = \int_{o}^{t_o} a\ t^b\ e^{-ct}\ dt + \int_{t_o}^{\infty} q_o\ e^{-\frac{t-t_o}{d}}\ dt \qquad [6]$$

Q is the total volume of runoff in the hydrograph expressed in cubic feet per second hours; t_o is equal to kt_p, the product of a factor and time to peak; q_o is the discharge in cubic feet per second at the time when recession begins (t_o); t is the time in hours from the beginning of the rise; a, b, and c are constants obtained by solution of the hydrograph equation; and d and k are constants obtained from analysis of hydrograph characteristics.

The first part of equation 6 expresses the form of the hydrograph through the peak to the point of recession. Therefore, equation 1

$$q = a\ t^b\ e^{-ct}$$

may be used to help solve for constants a, b, and c. The peak flow occurs where the first derivative of q with respect to t is zero.

$$\frac{dq}{dt} = a\ t^b\ (-c)\ e^{-ct} + a\ b\ t^{b-1}\ e^{-ct}$$

and by simplifying,

$$\frac{dq}{dt} = \frac{a}{e^{ct}}\ (-t^b\ c + b\ t^{b-1})$$

The maximum or minimum occurs where $\frac{dq}{dt}$ is equal to zero. Since $\frac{a}{e^{ct}}$ is nonzero, the maximum value of q occurs where:

$$t^b\ c = b\ t^{b-1}$$

or since t is always positive:

$$\frac{t^b}{t^{b-1}} = \frac{b}{c} = t$$

At this point $t = t_p$; therefore:

$$c = \frac{b}{t_p} \qquad [7]$$

where t_p is the time to peak in hours.

By substituting $t = t_p$ in equation 1, the equation for peak discharge is:

$$q_p = a\ t_p^{\,b}\ e^{-ct_p}$$

letting $c = \frac{b}{tp}$ and solving for a:

$$a = \frac{q_p\ e^b}{t_p^{\,b}} \qquad [8]$$

Thus, the constants a and c in equation 1 can be expressed in terms of b, t_p, and q_p.

Next, q_o for use in equation 5 is determined by setting $t = t_o$ in equation 1:

$$q_o = a\ t_o^{\,b}\ e^{-ct_o} \qquad [9]$$

By substituting $t_o = kt_p$ and equations 7 and 8 into equation 9:

$$q_o = \frac{q_p\ e^b}{t_p^{\,b}}\ (kt_p)^b\ e^{-\frac{b\ kt_p}{t_p}}$$

and simplifies to:

$$q_o = q_p\ k^b\ e^b\ e^{-bk} \qquad [10]$$

By substituting $t_o = kt_p$ and equations 7, 8, and 10 into equation 6, the total volume of runoff in terms of b is:

$$Q = \int_o^{kt_p} \frac{q_p\ e^b}{t_p^{\,b}}\ t^b\ e^{-\frac{bt}{t_p}}\ dt$$

$$+ \int_{kt_p}^{\infty} q_p\ k^b\ e^b\ e^{-bk}\ e^{-\frac{t-kt_p}{d}}\ dt \qquad [11]$$

Solution of equation 11 is simplified if time is transformed such that t_p is unity. Therefore, letting $t_1 = \frac{t}{t_p}$, when $t = t_p$, $t_{1_p} = 1.0$, $t = t_1 t_p$, and $dt = t_p\, dt_1$; equation 11 becomes:

$$Q = \int_0^{kt_{1_p}} \frac{q_p\, e^b}{t_p{}^b} (t_1 t_p)^b\, e^{-\frac{bt_1 t_p}{t_p}}\, t_p\, dt_1$$

$$+ \int_{kt_{1_p}}^{\infty} q_p\, k^b\, e^b\, e^{-bk}\, e^{-\frac{t_1 t_p - kt_p}{d}}\, t_p\, dt_1$$

By simplifying and using $t_{1_p} = 1.0$:

$$Q = q_p\, e^b\, t_p \int_0^k t_1{}^b\, e^{-bt_1}\, dt_1$$

$$+ q_p\, e^b\, k^b\, e^{-bk}\, t_p \int_k^{\infty} e^{-\frac{t_1 t_p - kt_p}{d}}\, dt_1$$

By dividing both sides by $q_p t_p$ and solving the integrals:

$$\frac{Q}{q_p t_p} = \left[e^b \left[e^{-bt_1} \left[\frac{t_1{}^b}{-b} - \frac{bt_1{}^{b-1}}{(-b)^2} + \frac{b(b-1)t_1{}^{b-2}}{(-b)^3} - \cdots \right.\right.\right.$$

$$\left.\left.\left. + (-1)^{b-1} \frac{b!\, t_1}{(-b)^b} + (-1)^b \frac{b!}{(-b)^{b+1}} \right] \right]_0^k \Bigg]$$

$$+ k^b\, e^b\, e^{-bk}\, e^{\frac{t_p k}{d}} \left[-\frac{d}{t_p}\, e^{-\frac{t_1 t_p}{d}} \right]_k^{\infty}$$

or by simplifying, the second half of the equation becomes:

$$\frac{Q}{q_p t_p} = \left[e^b \left[e^{-bt_1} \left[\frac{t_1^b}{-b} - \frac{bt_1^{b-1}}{(-b)^2} + \frac{b(b-1)t_1^{b-2}}{(-b)^3} - \cdots \right. \right. \right.$$

$$\left. \left. \left. + (-1)^{b-1} \frac{b!t_1}{(-b)^b} + (-1)^b \frac{b!}{(-b)^{b+1}} \right] \right]_o^k \Bigg] + \frac{d}{t_p} k^b e^{b(1-k)} \qquad [12]$$

In the above equation all things are known except b; Q is the volume of runoff; q_p is the instantaneous peak runoff rate; t_p is the time to peak; d is the average recession constant for the watershed; and k is the ratio of the time at which the recession begins to the time to peak.

Since there is no direct solution of equation 12 for b, a graphical solution is necessary. The graphical solution would be a family of curves in which $\frac{Q}{q_p t_p}$ and $\frac{d}{t_p}$ are known parameters. Such a family of curves would be needed for each value of k. For practical purposes, it was assumed that one family of curves based on a constant value of k would be satisfactory. In order to establish the factor k, an average value for the six watersheds, about 1.8, was used as the first trial value. The value of 1.8 tended to cause the recession to start too late even though it was fairly close to all actual values. This may be partially explained by the fact that values of the equation $q = a\, t^b\, e^{-ct}$ are falling considerably faster on the recession side of the hydrograph than are the values of the actual hydrograph. For this reason, the equation $q = q_o\, e^{-\frac{t-t_o}{d}}$ becomes effective at a lower value of q_o than it should. Further analysis indicated that this is partially corrected by assuming that the recession begins at an earlier period, k equal to about 1.5. The value of 1.5 appeared to be satisfactory for all six watersheds. On this basis, it was assumed that one family of curves based on a k value of 1.5 would be satisfactory for all cases. Figure 2 shows the hydrograph of Rock Creek and two equations obtained by using values of 1.5 and 1.8 for k.

By substituting the value 1.5 for k in equation 12, the equation is:

$$\frac{Q}{q_p t_p} = e^b \left[e^{-bt_1} \left[\frac{t_1^b}{-b} - \frac{bt_1^{b-1}}{(-b)^2} + \frac{b(b-1)t_1^{b-2}}{(-b)^3} - \cdots \right.\right.$$

$$\left.\left. + (-1)^{b-1} \frac{b!t_1}{(-b)^b} + (-1)^b \frac{b!}{(-b)^{b+1}} \right] \right]_0^{1.5} + \frac{d}{t_p} (1.5)^b e^{-0.5\,b} \qquad [13]$$

The family of curves for solution of equation 13 is shown in figure 3. $\frac{Q}{q_p t_p}$ was plotted versus b for various values of $\frac{d}{t_p}$.

For constant values of k, equation 13 may be shown as:

$$\frac{Q}{q_p t_p} = f(b) + \frac{d}{t_p} g(b) \qquad [14]$$

In case the ranges of $\frac{Q}{q_p t_p}$, $\frac{d}{t_p}$, or b, shown in figure 3, are not wide enough, values of f(b) and g(b) for k = 1.5 and b = 0 to 40 are presented in table 2. These values may be used to calculate the points required to plot a family of curves similar to that of figure 3. They may also be used in graphical interpolation for precise values of b by expanding the graph of $\frac{Q}{q_p t_p}$ versus b for a particular value of $\frac{d}{t_p}$. The set of four integer values of b are estimated from the family of curves such that they bracket the true value of b. Using these four values and $\frac{d}{t_p}$, values of $\frac{Q}{q_p t_p}$ are calculated and plotted versus b on a large scale. The points will plot as a smooth curve (almost linear). The exact value of $\frac{Q}{q_p t_p}$ is entered and the value of b is found.

Use of the hydrograph equation begins with solution of the variable b. Q, q_p, t_p, and d are determined for a hydrograph; b is then obtained from the curves of figure 3, or perhaps more accurately by using graphical interpolation with the data presented in table 2. Variables a and c are calculated from equations 8 and 7, respectively. With all the variables known, the discharge at any time can be calculated from equations 1 and 5. Equation 1 defines the hydrograph from time t equal to zero to t_o (1.5 times the time to peak). Equation 5 defines the recession of the hydrograph from time t_o to the end of the hydrograph. Before equation 5 may be used, the value of q_o at time t_o must be calculated from equation 1. Table 3 is a sample calculation of the hydrograph of Rock Creek. Figure 2 is a plot of the hydrograph and the equation for the Rock Creek watershed.

Figures 4 through 8 are the hydrographs and equations for the other five watersheds.

TABLE 1.--Major features of hydrographs for six tributaries of the Washita River, Okla.

Name	Drainage area	Volume	Time to peak	Peak rate	Rec. constant	Time constant
	Sq. miles	C.f.s. hr.	Hr.	C.f.s.	d	k
Barnitz Creek	243	156,815	21	9,500	4.82	1.52
Little Washita	227	146,490	14	14,000	6.11	1.57
Pond Creek	319	205,860	18	18,000	6.55	1.61
Rock Creek	138	89,056	7	13,000	5.89	1.86
Rush Creek	206	132,938	4	20,000	4.67	2.00
Sugar Creek	211	136,158	10	11,000	5.39	1.90

TABLE 2.--Values of f(b) and g(b) for use in equation 14

b	f(b)	g(b)	b	f(b)	g(b)
0	1,50000	1.00000	21	0.53188	0.13735
1	1.20196	0.90980	22	.52137	.12496
2	1.06552	.82773	23	.51140	.11369
3	0.97854	.75306	24	.50192	.10343
4	.91487	.68513	25	.49290	.09410
5	.86462	.62333	26	.48430	.08561
6	.82306	.56711	27	.47610	.07789
7	.78761	.51595	28	.46827	.07087
8	.75667	.46941	29	.46077	.06447
9	.72924	.42707	30	.45360	.05866
10	.70461	.38854	31	.44672	.05337
11	.68227	.35350	32	.44012	.04855
12	.66185	.32161	33	.43378	.04417
13	.64306	.29260	34	.42769	.04019
14	.62569	.26620	35	.42183	.03656
15	.60955	.24219	36	.41619	.03326
16	.59449	.22035	37	.41076	.03026
17	.58040	.20047	38	.40552	.02753
18	.56717	.18239	39	.40046	.02505
19	.55472	.16593	40	.39558	.02279
20	.54298	.15097			

TABLE 3.--Hydrograph calculation for Rock Creek

Equations used: $q = a\, t^b\, e^{-ct}$ $\quad q = q_o\, e^{-\frac{t-kt_p}{d}}$

Constants: $Q = 89{,}056$ $\quad d = 5.89$ $\quad t_p = 7$

$q_p = 13{,}000$ $\quad k = 1.5$ $\quad kt_p = 10.5$

$$\frac{Q}{q_p t_p} = \frac{89{,}056}{(7)(13{,}000)} = 0.9786 \qquad \frac{d}{t_p} = \frac{5.89}{7} = 0.8414$$

b = 11.00 from figure 2
= 11.027 by graphical interpolation

$$c = \frac{b}{t_p} = \frac{11.027}{7} = 1.57529$$

$$a = \frac{q_p\, e^b}{t_p{}^b} = \frac{(13{,}000)\, e^{11.027}}{7^{11.027}}$$

log a = log 13,000 + 11.027 log e - 11.027 log 7
= - 0.415993 = 9.584007 - 10.0
= 0.383713

$q = a\, t^b\, e^{-ct}$

log q = log a + b log t - ct log e
= - 0.415993 + 11.027 log t - 0.684139t

t	log t	log q	q
1	0	-1.100132	0
2	.301031	1.535187	34
3	.477121	2.792803	621
4	.602060	3.486367	3,065
5	.698970	3.870854	7,428
6	.778151	4.059844	11,477

Table 3.--Continued

t	log t	log q	q
7	.845098	4.113930	13,000
8	.903090	4.069268	11,730
9	.954243	3.949194	8,896
10	1.000000	3.769617	5,883
10.5	1.021189	3.661199	4,584

$$q = q_o \, e^{-\frac{t-kt_p}{d}}$$

$$\log q = \log q_o - \frac{t-kt_p}{d} \log e$$

$$= 3.661199 - \frac{t-10.5}{5.89} (0.434294)$$

$$= 4.435407 - 0.073734t$$

11	3.624333	4,211
12	3.550599	3,553
14	3.403131	2,530
16	3.255663	1,802
18	3.108195	1,283
20	2.960727	914
24	2.665791	463
28	2.370855	235
34	1.928451	85
40	1.486047	31
50	.748707	6

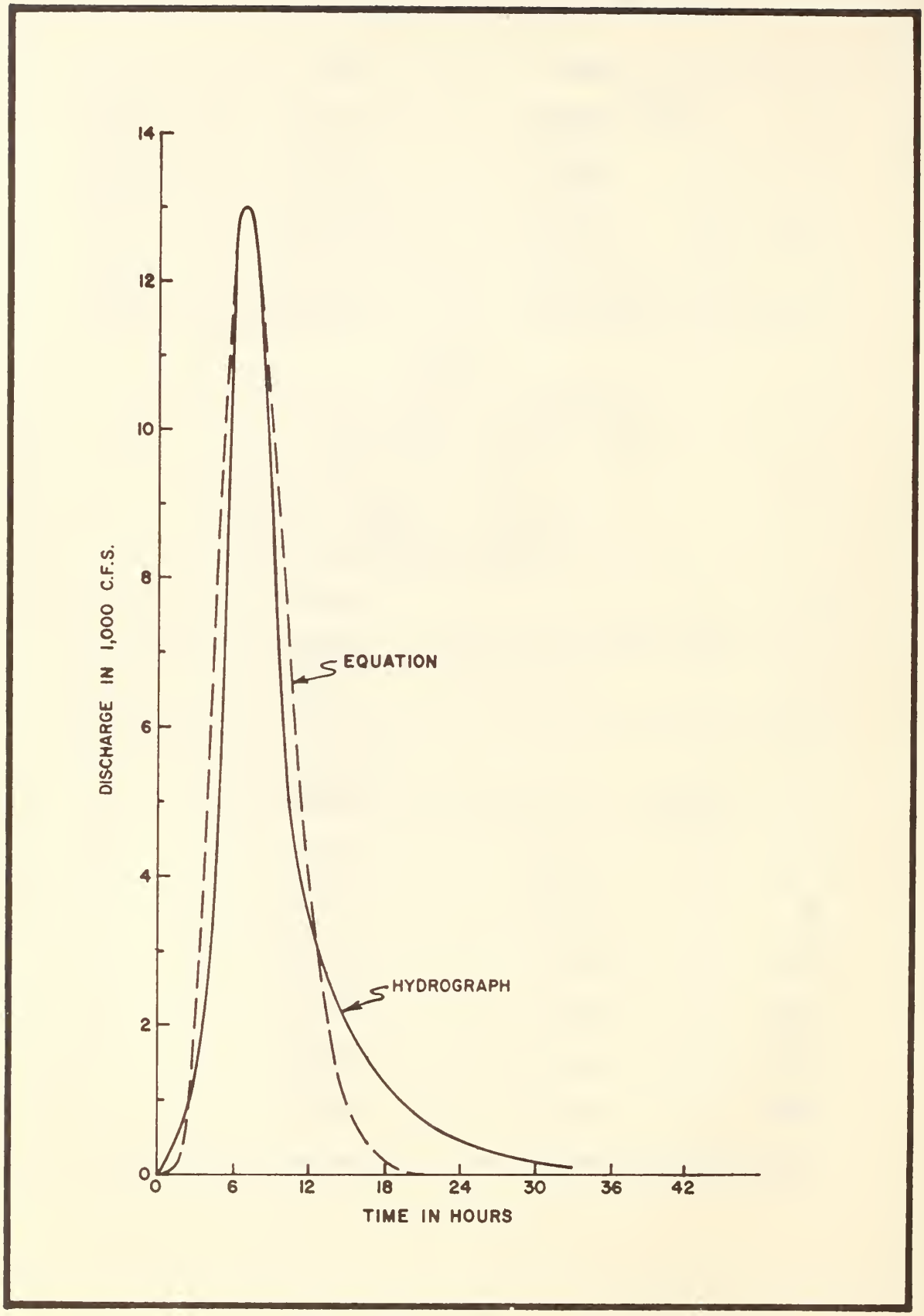

Figure 1.--Rock Creek hydrograph; equation $q = at^b e^{-ct}$.

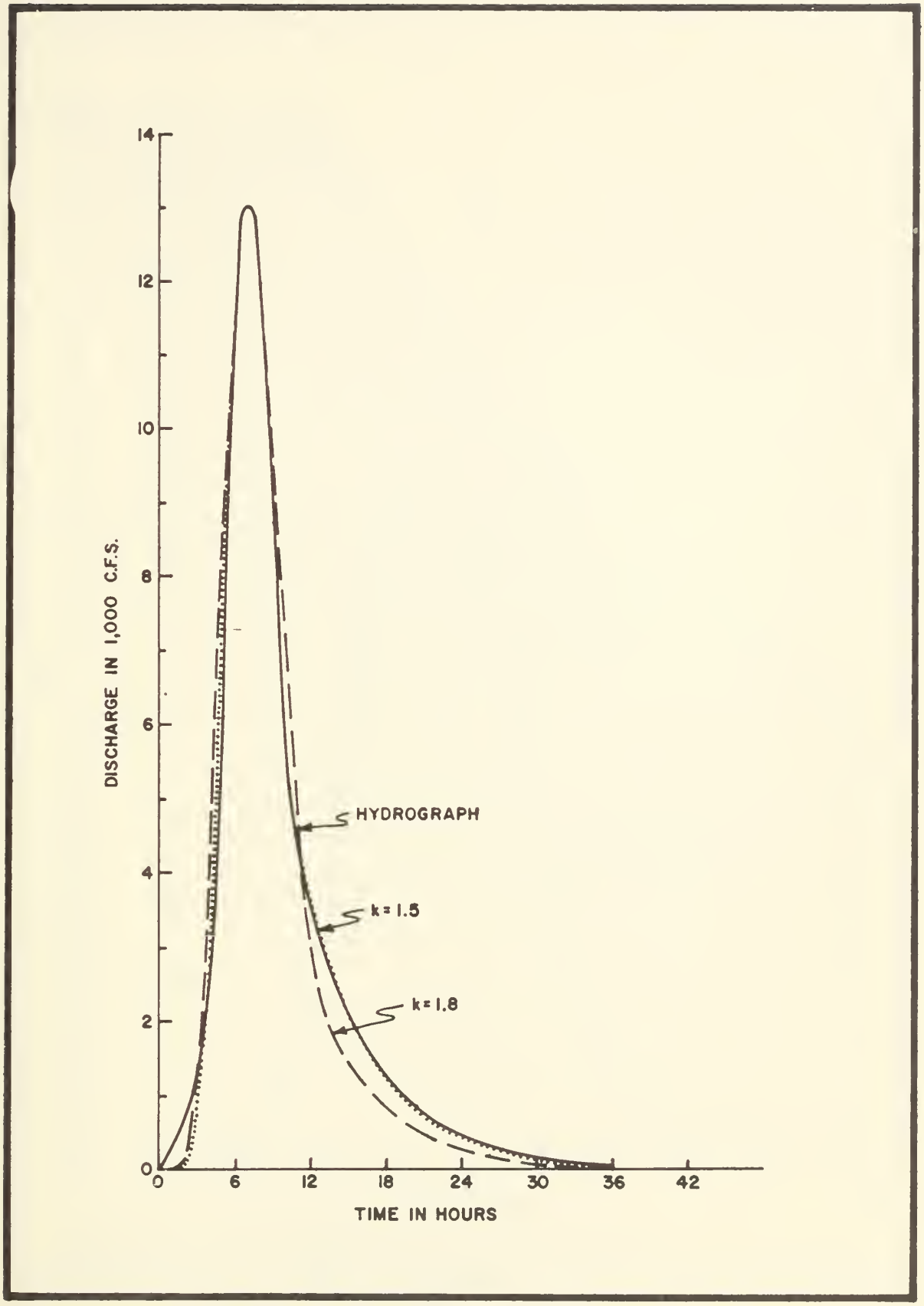

Figure 2.-- Rock Creek hydrograph.

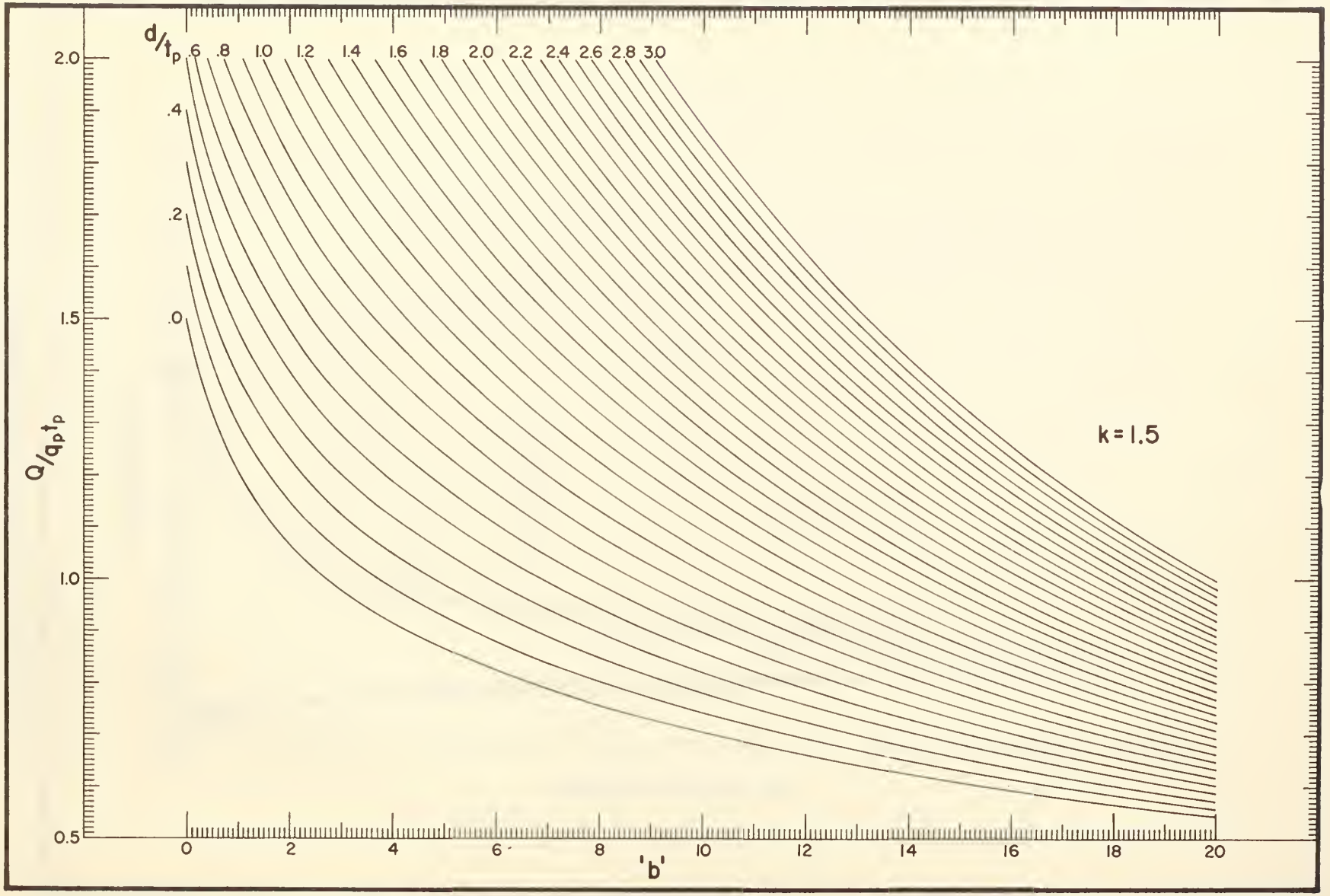

Figure 3. $O/q_p t_p$ Versus b for Various Values of d/t_p

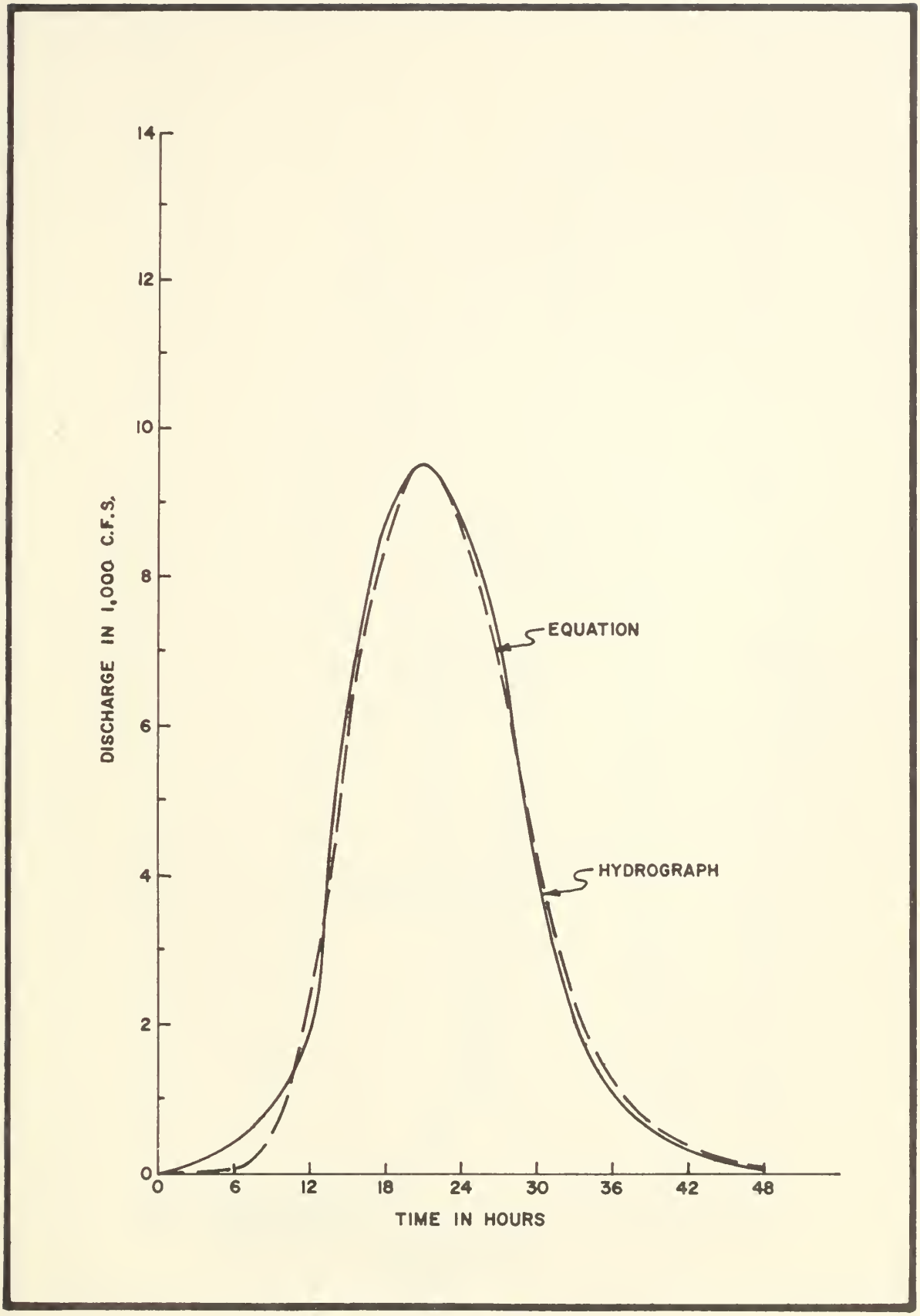

Figure 4.--Barnitz Creek hydrograph.

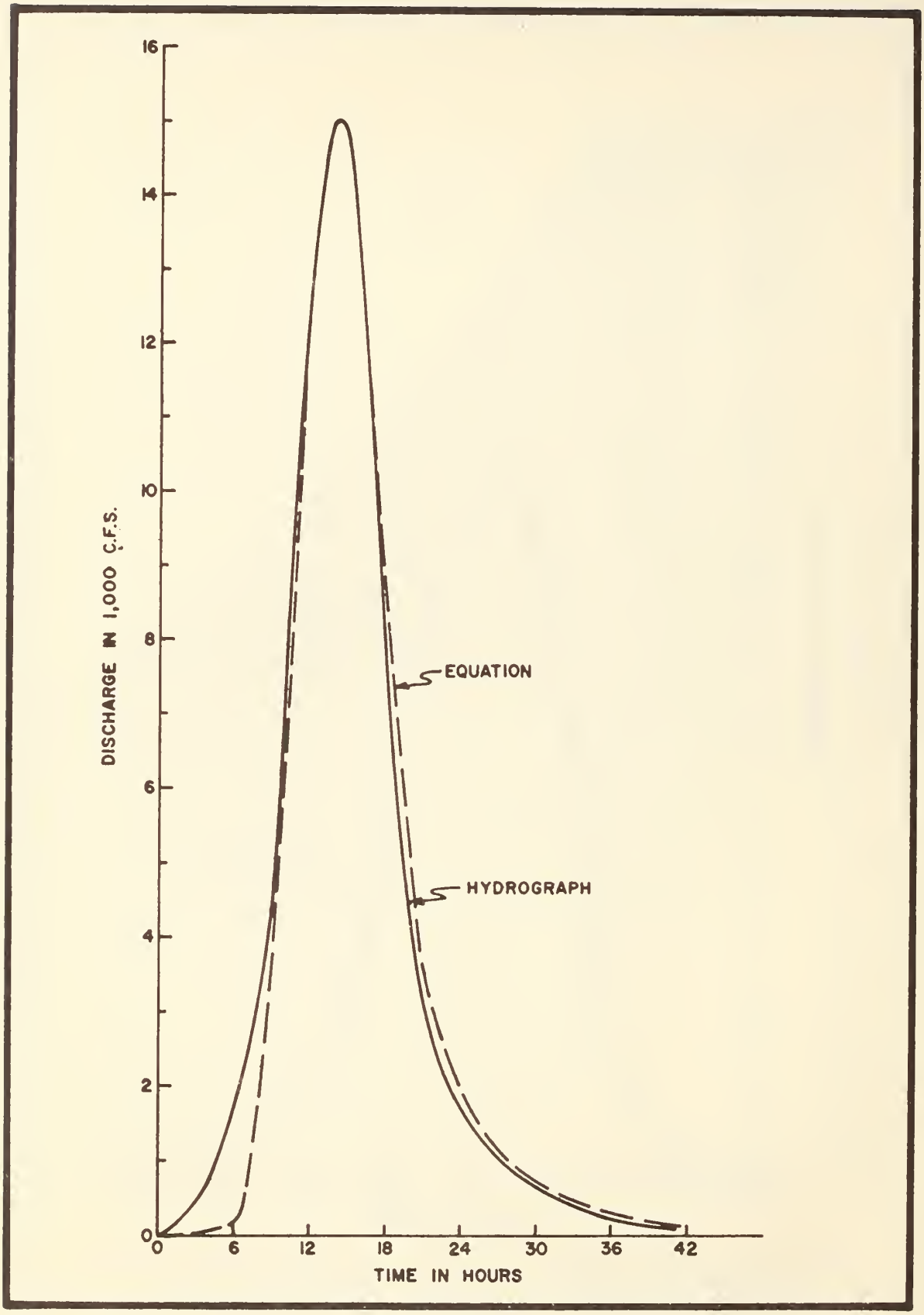

Figure 5.--Little Washita River hydrograph.

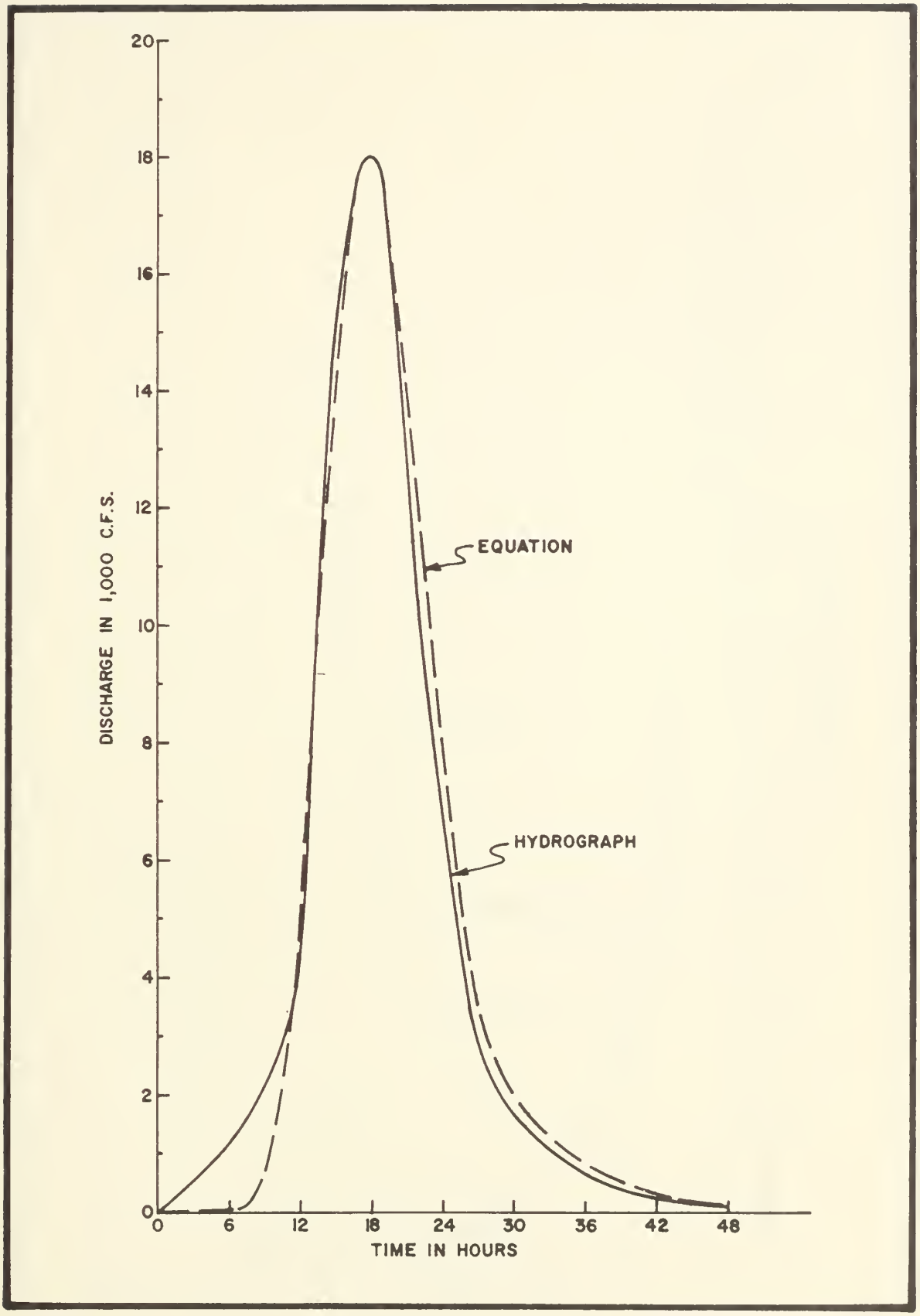

Figure 6.--Pond Creek hydrograph.

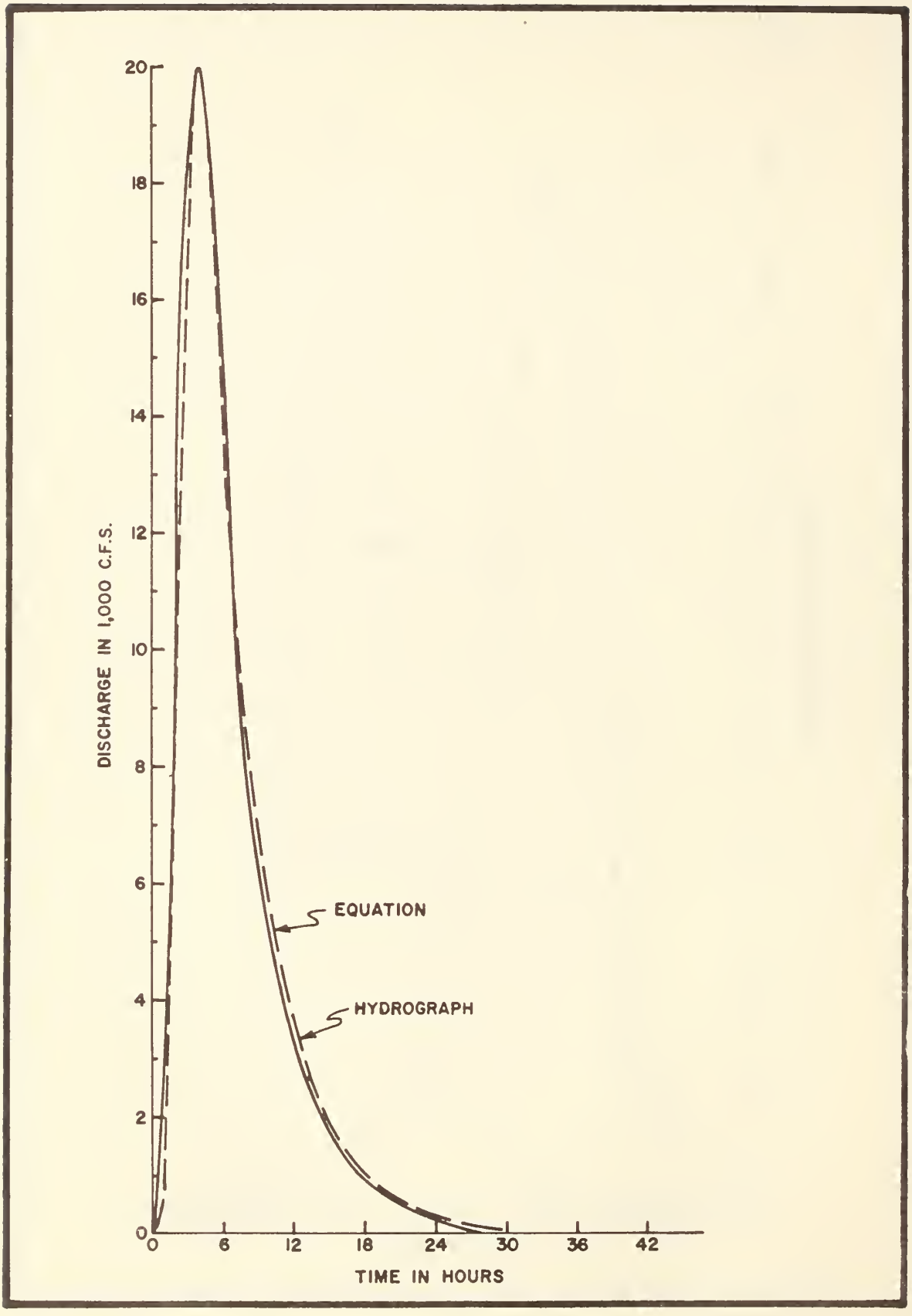

Figure 7.--Rush Creek Hydrograph.

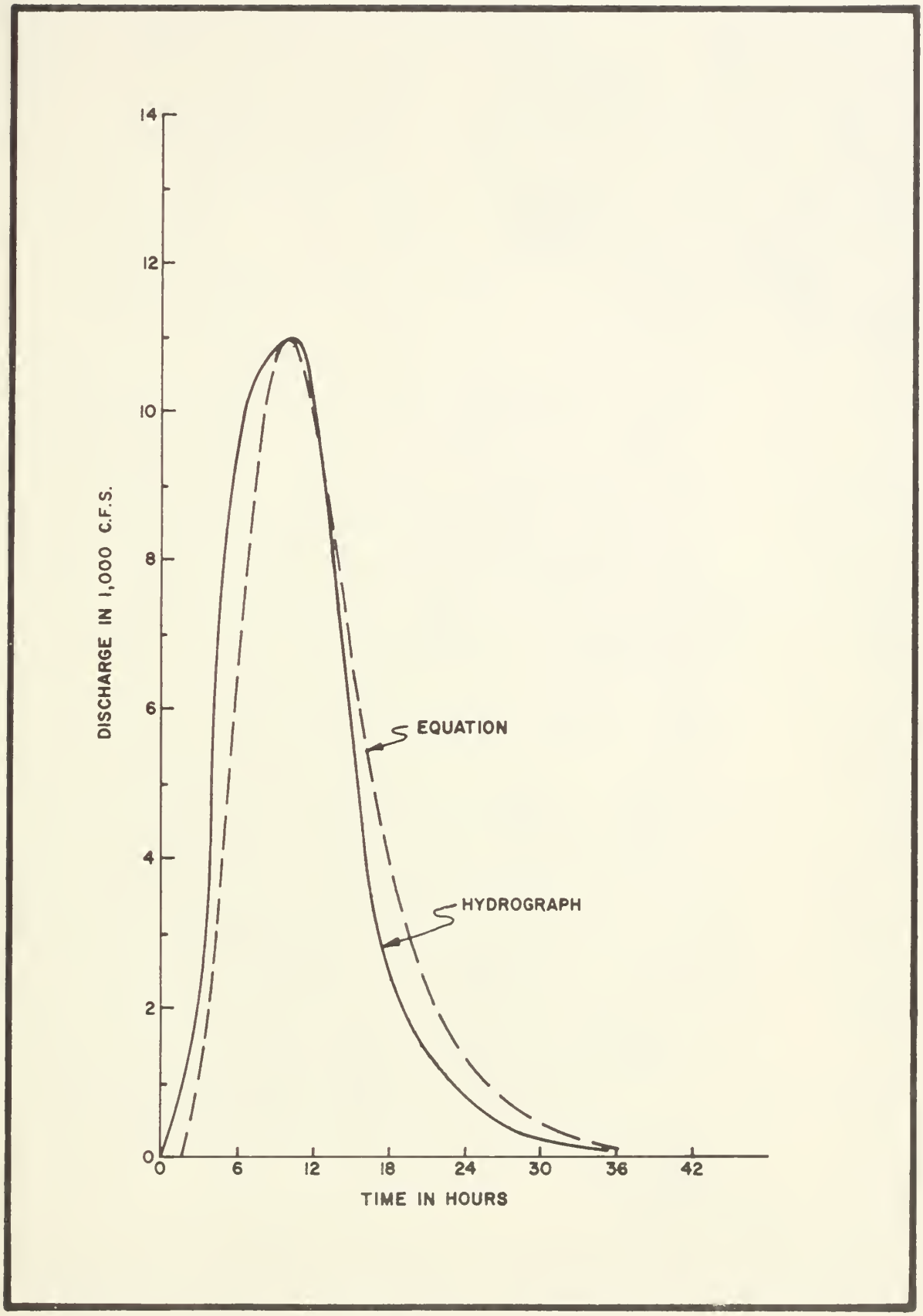

Figure 8.--Sugar Creek hydrograph.